ECE 1508S2: Applied Deep Learning

Chapter 8: Auto Encoders

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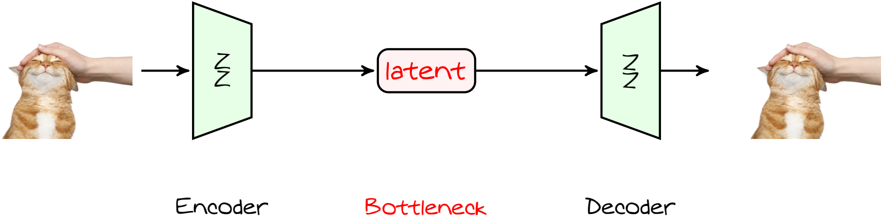
Auto Encoder: *Deep PCA with Encoder-Decoder*

AE is in principle a deep encoder-decoder architecture used for nonlinear PCA



Vanilla AE finds a latent space that is very smaller in dimension

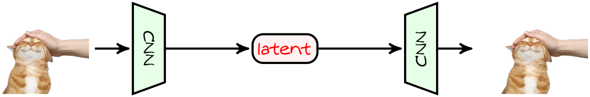
- *Each data-point is encoded to its low-dimensional latent representation*
  ↳ *Latent representation contains in fact the principle features of data*
- *Latent representation can re-generate the data-point via using decoder*

Vanilla AE

Auto Encoder (AE)

AE is an encoder-decoder architecture whose bottleneck feature, also called latent representation, has lower dimension than the input and output of NN

We can implement encoder and decoder by simple NNs, e.g.,



Such an architecture is a vanilla AE mainly used for compression

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+ *Is compression so crucial that AEs become so important?*
– Naive answer: Yes! Better answer: *AEs can do much more than compression in fact!*

Training AEs: *General Approach*

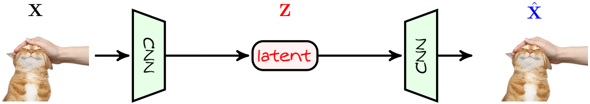
We typically use AEs in unsupervised settings: *it means that we have no labels in the dataset*

- *In AE both latent representation and decoded data are outputs*
- *For training we need to compute loss between the outputs and a reference*
  - ↳ *We cannot compare the latent representation with any reference*
    - ↳ *We have no true latent representation*
- *We should extract some reference from our dataset*
  - ↳ *This reference depends on our target application*
  - ↳ *We are going to consider three types in this chapter*

To go on with the training of AEs, let's keep the track of their applications

1. *Compression*
   - ↳ *We intend to compress data into a lower-dimensional subspace*
   - ↳ *For loss computation, we compare decoded data with its ground truth*

Training AEs *for Compression*



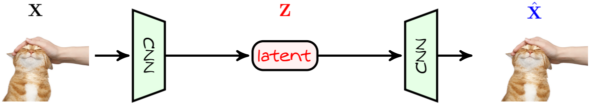
Let's name variables: *say the input is* $\mathbf{X}$*, e.g., RGB image, latent representation is* $\mathbf{Z}$*, e.g., multi-channel tensor, and* $\hat{\mathbf{X}}$ *is decoded output, e.g., RGB image*

- *For compression we wish to recover* $\hat{\mathbf{X}} = \mathbf{X}$
  - ↳ *Loss is proportional to the difference between* $\mathbf{X}$ *and* $\hat{\mathbf{X}}$
- *We are indifferent about the behavior of latent representation*
  - ↳ *We do not need to include* $\mathbf{Z}$ *directly in loss computation*
  - ↳ $\mathbf{Z}$ *contributes to loss indirectly through* $\hat{\mathbf{X}}$

So, the loss in this case is compute as

$$\hat{R} = \mathcal{L}(\hat{\mathbf{X}}, \mathbf{X})$$

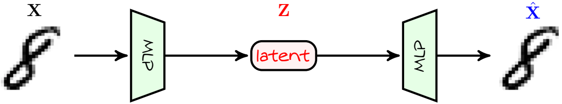
Training AEs *for Compression*



We know the loss: *we can analytically compute $\nabla_{\hat{\mathbf{x}}} \hat{R}$, so training is done by standard forward and backward pass*

1. *Pass $\mathbf{X}$ forward through the encoder*
   - ↳ *Compute output of all layer as well as $\mathbf{Z}$*
2. *Pass $\mathbf{Z}$ forward through the decoder*
   - ↳ *Compute output of all layer as well as $\hat{\mathbf{X}}$*
3. *Compute $\nabla_{\hat{\mathbf{x}}} \hat{R}$ and backpropagate through decoder*
4. *Compute $\nabla_{\mathbf{z}} \hat{R}$ from the gradient at the first layer of decoder*
5. *Starting from $\nabla_{\mathbf{z}} \hat{R}$ backpropagate through encoder*
6. *Update all weights and go for the next round*

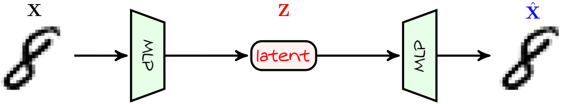
Example: *Compressing MNIST*



A simple practice can be done on MNIST: *we try to represent MNIST images in a 2-dimensional latent space*. *For encoding we use the following MLP*

1. *It has four hidden layer*
   - ↳ *The widths of layers gradually reduce*
   - ↳ *The last layer has only two outputs*
2. *All neurons are activated via sigmoid*
3. *We do not use any dropout or batch-normalization*

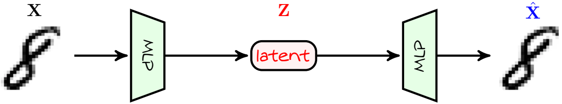
Example: *Compressing MNIST*



For decoding we use another MLP to invert the encoder

1. *The decoder has four hidden layer*
   - ↳ *The widths of layers gradually increase*
   - ↳ *The last layer has 784 neurons*
2. *All neurons are activated via sigmoid*
3. *We finally sort the output into a 28×28 matrix*

Example: *Compressing MNIST*



Training then follows the standard approach: this is in fact an 8-layer MLP

1. *Pass each training image forward through all layers*
2. *Compute loss between the output and true image, e.g.,*

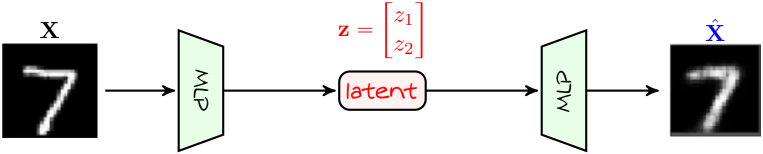
$$\mathcal{L}\left(\hat{\mathbf{X}}, \mathbf{X}\right) = \|\hat{\mathbf{X}} - \mathbf{X}\|^2$$

3. *Compute $\nabla_{\hat{\mathbf{X}}} \hat{R}$ and backpropagate*
4. *Update all weights and go for the next round*

Example: *Compressing MNIST*

We can then test our AE

1. *Pass a test image forward through encoder*
2. *Compute latent representation*
3. *Pass the latent representation forward through decoder*
4. *Compare the images*



Obvious Compression via Vanilla AE

For compression it is important that we set

the latent representation to be of smaller dimension than input

If we set it larger or equal to the input size, we end up with an obvious solution

$$\text{Decoder}\left(\text{Encoder}\left(\cdot\right)\right) = \text{Identity}\left(\cdot\right)$$

- *We want to recover the original data after decoding*
  - ↳ *Identity is always an obvious solution*
- *With larger latent space we can always realize identity*
  - ↳ *We simply set* $\mathbf{Z} = \mathbf{X}$ *and* $\hat{\mathbf{X}} = \mathbf{Z}$
- *This is however useless since we do not compress*

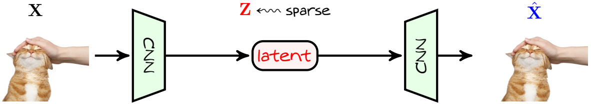
Sparse AEs

Let's keep the track of their applications

1. *Compression*
2. *Finding a sparse representation of data*
    - ↳ *We intend to represent data with a sparse object*
        - ↳ *for instance, we intend to represent input $\mathbf{x} \in \mathbb{R}^{100}$ with another 100-dimensional vector whosemost of entries are zero*
        - ↳ *we may want to further compress, i.e., represent $\mathbf{x} \in \mathbb{R}^{100}$ with an 80-dimensional sparse vector*
    - ↳ *For loss computation, we should also take a look at the latent representation*
        - ↳ *we want the latent representation to be sparse*
        - ↳ *in vanilla AE there is no guarantee that this happens*

For such application we use sparse AEs

Training AEs *for Sparse Representation*



Let's formulate the problem: *say the input is* $\mathbf{X}$, *latent representation* is $\mathbf{Z}$, *and* $\hat{\mathbf{X}}$ *is decoded output*

- *We still need to recover from latent representation, i.e., we want* $\hat{\mathbf{X}} = \mathbf{X}$
  - ↳ *Loss is proportional to the difference between* $\mathbf{X}$ *and* $\hat{\mathbf{X}}$
- *We also want to have sparse latent representation*
  - ↳ $\mathbf{Z}$ *should contribute directly to loss*
  - ↳ *Loss should also be proportional to the sparsity of* $\mathbf{Z}$

Training Sparse AEs

Loss is proportional to difference between $\mathbf{X}$ and $\hat{\mathbf{X}}$, and sparsity of $\mathbf{Z}$

So, the loss in this case should be

$$\hat{R} = \mathcal{L}(\hat{\mathbf{X}}, \mathbf{X}) + \lambda S(\mathbf{Z})$$

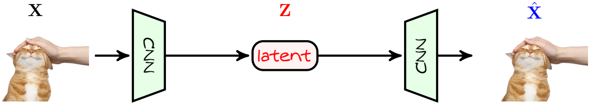
for some function $S(\cdot)$ that is proportional to sparsity, i.e.,

if $\mathbf{Z}$ has less zeros $\rightsquigarrow$ $S(\mathbf{Z})$ should increase

and regularizer λ that is a hyperparameter

- $S(\mathbf{Z}) = \|\mathbf{Z}\|_0 \rightsquigarrow$ *non-differentiable* ✗
- $S(\mathbf{Z}) = \|\mathbf{Z}\|_1 \rightsquigarrow$ *convex* ✓
- $S(\mathbf{Z}) = \mathrm{KL}(p_{\mathbf{Z}} \| \mathrm{Ber}_\rho) \rightsquigarrow$ *convex* ✓
  - ↳ Ber_ρ *is a Bernoulli distribution with probability of zero being ρ*
  - ↳ $p_{\mathbf{Z}}$ *is the empirical distribution of the support of $\mathbf{Z}$*

Training Sparse AEs



Let's see how training looks: *say we are training with single sample* $\mathbf{X}$

- *Pass forward* $\mathbf{X}$ *through encoder and decoder*
- *Backpropagate by first computing* $\nabla_{\hat{\mathbf{X}}} \hat{R}$
  - ↳ *Backpropagate till the bottleneck*
  - ↳ *At the bottleneck, we need to compute* $\nabla_{\mathbf{Z}} \hat{R}$

$$\nabla_{\mathbf{Z}} \hat{R} = \underbrace{\nabla_{\hat{\mathbf{X}}} \hat{R} \circ \nabla_{\mathbf{Z}} \hat{\mathbf{X}}}_{\text{computed by backpropagation}} + \lambda \nabla_{\mathbf{Z}} S\left(\mathbf{Z}\right)$$

  - ↳ *Start from* $\nabla_{\mathbf{Z}} \hat{R}$ *and backpropagate till input*
- *Update weights and go for the next round*

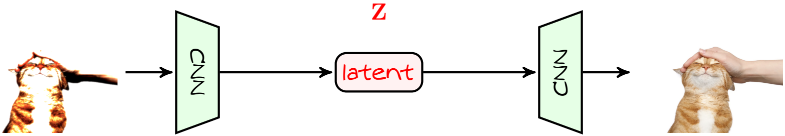
Using AE for Noise Removal

Let's keep the track of their applications

1. *Compression*
2. *Finding a sparse representation of data*
3. *Denoising*
   - ↳ *We intend to find a representation that can refine noisy data*
     - ↳ *for instance we want to remove background noise from an image*
     - ↳ *for instance we want to increase the resolution of an image*
     - ↳ *for instance we want to color a gray image*
   - ↳ *For loss computation, we should*
     - ↳ *be able the recover the refined data at the decoder*
     - ↳ *unlike vanilla AE we can start with distorted data*

We call such AE architectures denoising AEs

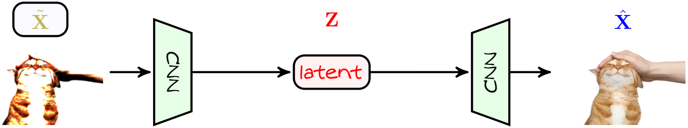
Training Denoising AEs



We can train a denoising AE using *degraded samples*

- *For each training sample we generate its degraded counterpart, e.g.,*
  ↳ *for each image we also produce a noisy, or low-resolution or gray version*
- *We give this noisy version to the encoder*
- *We set loss to compute difference between original samples and output*
  ↳ *the decoded image and the original RGB image in dataset*

Training Denoising AEs



Let's formulate the problem: *say the sample is $\mathbf{X}$, and its corrupted version is $\tilde{\mathbf{X}}$. Also, denote latent representation by $\mathbf{Z}$ and decoded output by $\hat{\mathbf{X}}$*

- *We want to recover original data from latent representation, i.e., $\hat{\mathbf{X}} = \mathbf{X}$*
  ↳ *Loss is proportional to the difference between $\mathbf{X}$ and $\hat{\mathbf{X}}$*
- *We may want our representation to be sparse*
  ↳ *We could add a penalty proportional to sparsity of $\mathbf{Z}$*

So, we set the loss to

$$\hat{R} = \mathcal{L}(\hat{\mathbf{X}}, \mathbf{X}) + \lambda S(\mathbf{Z})$$

Training AEs: *Summary*

We could have various form of AEs depending on the target application

- *Vanilla* AEs
  - ↳ *Encoder-decoder with both input and label being data*
  - ↳ *Loss computes difference between input and output* $\equiv$ *recovery error*
  - ↳ *We can use it for compression*
- *Sparse* AEs
  - ↳ *Encoder-decoder with both input and label being data*
  - ↳ *Loss computes recovery error plus a sparsity penalty*
  - ↳ *We can use it for sparse representation of data*
- *Denoising* AEs
  - ↳ *Encoder-decoder with input being noisy data and label being data*
  - ↳ *Loss computes recovery error*
  - ↳ *We can also regularize with a sparsity penalty if we need sparse latent*
  - ↳ *We can use it for noise removal, resolution increasing and other similar applications*