# Applied Deep Learning

## Chapter 6: Recurrent NNs

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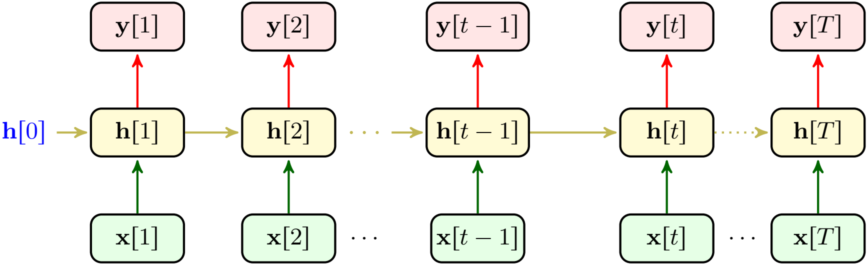
# Training our Shallow RNN

*We now want to train this basic RNN*




*Let's consider training for only one sample*

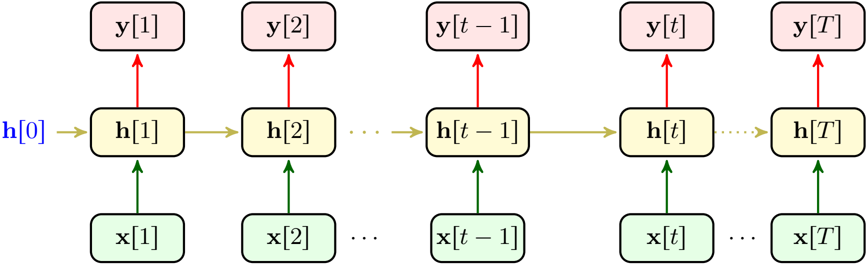
*We assume that we have a sequence of labels* $\mathbf{v}[1], \ldots, \mathbf{v}[T]$

1. *We know some $\mathbf{v}[t]$ could be empty, e.g., in many-to-one scenario*
2. *So could be some of $\mathbf{x}[t]$'s, e.g., in one-to-many scenario*

*We however have no problem with that!*

# Training our Shallow RNN

*We now want to train this basic RNN*




*Let's consider training for only one sample*

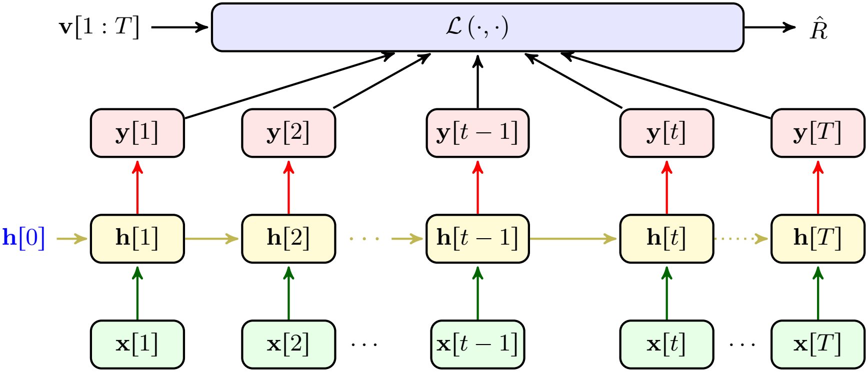
*The loss in general can be written as*

$$\hat{R} = \mathcal{L}\left(\mathbf{y}[1:T], \mathbf{v}[1:T]\right)$$

*where we use shorten notation* $\mathbf{y}[1:T] = \mathbf{y}[1], \dots, \mathbf{y}[T]$

# Training our Shallow RNN

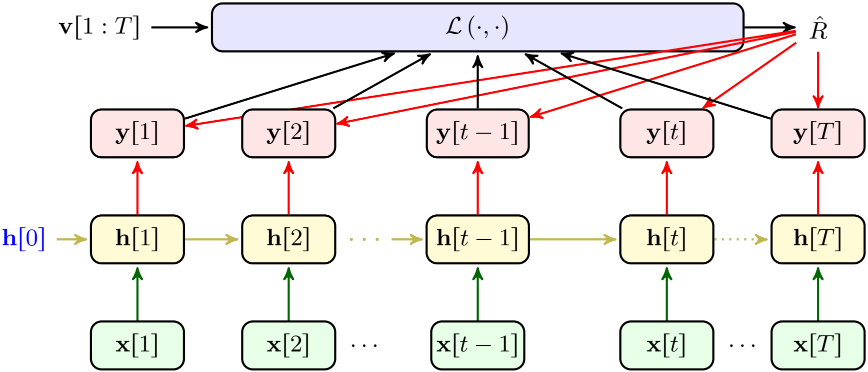
*We can think of such a diagram*




+ *It seems to be hard to get back from $\hat{R}$ to each $\mathbf{y}[t]$*
– Well! That's true, but we have some remedy for it

# Training our Shallow RNN

*We can think of such a diagram*




*For the moment, we assume that we can*

*compute the* $\nabla_{\mathbf{y}[t]}\hat{R}$ *for all* $t \equiv$ *move backward from* $\hat{R}$ *to any* $\mathbf{y}[t]$

# Backward Pass Through Time

Starting from $\hat{R}$, *say we want to find* $\nabla_{\mathbf{W}_{\mathrm{m}}} \hat{R}$

- *Since* $\hat{R}$ *is a function of* $\mathbf{y}[1:T]$, *we should write a vectorized chain rule*

$$\begin{aligned}\nabla_{\mathbf{W}_{\mathrm{m}}} \hat{R} &= \nabla_{\mathbf{y}[1]} \hat{R} \circ \nabla_{\mathbf{W}_{\mathrm{m}}} \mathbf{y}[1] + \ldots + \nabla_{\mathbf{y}[T]} \hat{R} \circ \nabla_{\mathbf{W}_{\mathrm{m}}} \mathbf{y}[T] \\ &= \sum_{t=1}^{T} \nabla_{\mathbf{y}[t]} \hat{R} \circ \nabla_{\mathbf{W}_{\mathrm{m}}} \mathbf{y}[t]\end{aligned}$$

- *Since we assumed that we have* $\nabla_{\mathbf{y}[t]} \hat{R}$, *our main task is to find*

$$\nabla_{\mathbf{W}_{\mathrm{m}}} \mathbf{y}[t]$$

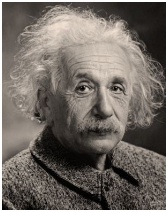
  *for all* $t$: *we could hence compute it for a general* $t$

  ↳ *This is a tensor-like gradient, i.e.,* $[\nabla_{\mathbf{W}_{\mathrm{m}}} y_1[t], \ldots, \nabla_{\mathbf{W}_{\mathrm{m}}} y_M[t]]$

- *Apparently, we should apply chain rule for several times!*

# Backward Pass Through Time

+ *But, this is going to be exhausting?*
– Well, we could again follow Albert Einstein advice!



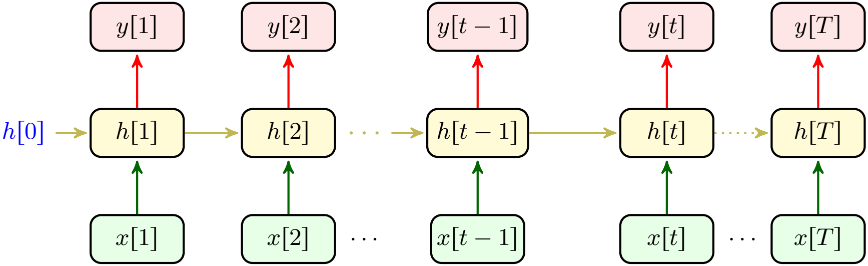
*"Everything should be made as simple as possible, but not simpler!"*

# Backward Pass Through Time: *Simple Example*

*Let's consider a dummy RNN with all variables being scalar*

1. *We have* $y[t] = f(w_2 h[t])$
2. *We have* $h[t] = f(w_1 x[t] + w_{\mathrm{m}} h[t-1])$
3. *We start with hidden state* $h[0]$

*So the diagram gets simplified as below*



# Backward Pass Through Time: *Simple Example*

Starting from $\hat{R}$, say we want to find $\nabla_{w_{\rm m}} \hat{R}$: *our main task is to find*

$$\frac{\partial y[t]}{\partial w_{\rm m}}$$

*Let's start the computation: say we have passed forward through the RNN*

- *we now have $y[t]$, $h[t]$ and $x[t]$ for all $t$*

*To go backward, we note that $y[t] = f(w_2 h[t])$*

- *$y[t]$ is a function of $h[t]$: so we write the chain rule as*

$$\frac{\partial y[t]}{\partial w_{\rm m}} = \frac{\partial y[t]}{\partial h[t]} \frac{\partial h[t]}{\partial w_{\rm m}}$$

↳ *we can compute the first term as $\partial y[t] / \partial h[t] = w_2 \dot{f}(w_2 h[t])$*
↳ *we should compute the second term by further chain rule*

# Backward Pass Through Time: *Simple Example*

$$\frac{\partial y[t]}{\partial w_{\mathrm{m}}} = w_2 \dot{f}(w_2 h[t]) \frac{\partial h[t]}{\partial w_{\mathrm{m}}}$$

*We keep going backward, by noting that* $h[t] = f(w_1 x[t] + w_{\mathrm{m}} h[t-1])$

- $h[t]$ *is a function of* $w_{\mathrm{m}}$ *and* $h[t-1]$*:*[1]
  ↳ *let's write* $h[t] = g\left(w_{\mathrm{m}}, h[t-1]\right)$
- *So we write the chain rule as*

$$\begin{aligned}\frac{\partial h[t]}{\partial w_{\mathrm{m}}} &= \frac{\partial g}{\partial w_{\mathrm{m}}} \underbrace{\frac{\partial w_{\mathrm{m}}}{\partial w_{\mathrm{m}}}}_{1} + \frac{\partial g}{\partial h[t-1]} \frac{\partial h[t-1]}{\partial w_{\mathrm{m}}} \\ &= \frac{\partial g}{\partial w_{\mathrm{m}}} + \frac{\partial g}{\partial h[t-1]} \frac{\partial h[t-1]}{\partial w_{\mathrm{m}}}\end{aligned}$$

[1] *We ignore* $w_1$ *and* $x[t]$*, as they are obviously not functions of* $w_{\mathrm{m}}$

# Backward Pass Through Time: *Simple Example*

$$\frac{\partial y[t]}{\partial w_{\mathrm{m}}} = w_2 \dot{f}(w_2 h[t]) \frac{\partial h[t]}{\partial w_{\mathrm{m}}}$$

*We keep going backward, by noting that* $h[t] = f(z[t])$

- *Let's define* $z[t] = w_1 x[t] + w_{\mathrm{m}} h[t-1]$
- $h[t] = g\left(w_{\mathrm{m}}, h[t-1]\right)$
  - ↳ *we can compute* $\partial g / \partial w_{\mathrm{m}} = h[t-1] \dot{f}(z[t])$
  - ↳ *we can compute* $\partial g / \partial h[t-1] = w_{\mathrm{m}} \dot{f}(z[t])$
- *So we can simplify the chain rule as*

$$\begin{aligned} \frac{\partial h[t]}{\partial w_{\mathrm{m}}} &= h[t-1] \dot{f}\left(z[t]\right) + w_{\mathrm{m}} \dot{f}\left(z[t]\right) \frac{\partial h[t-1]}{\partial w_{\mathrm{m}}} \\ &= \dot{f}\left(z[t]\right) \left( h[t-1] + w_{\mathrm{m}} \frac{\partial h[t-1]}{\partial w_{\mathrm{m}}} \right) \end{aligned}$$

# Backward Pass Through Time: *Simple Example*

$$\frac{\partial y[t]}{\partial w_{\mathrm{m}}} = w_2 \dot{f}(w_2 h[t]) \dot{f}\left(z[t]\right) \left( h[t-1] + w_{\mathrm{m}} \frac{\partial h[t-1]}{\partial w_{\mathrm{m}}} \right)$$

*We keep going backward:* $h[t-1] = f(z[t-1])$

↳ *Recall that* $z[t-1] = w_1 x[t-1] + w_{\mathrm{m}} h[t-2]$

- *We just need to replace* $t$ *with* $t-1$ *in the last derivation*

$$\frac{\partial h[t-1]}{\partial w_{\mathrm{m}}} = \dot{f}\left(z[t-1]\right) \left( h[t-2] + w_{\mathrm{m}} \frac{\partial h[t-2]}{\partial w_{\mathrm{m}}} \right)$$

# Backward Pass Through Time: *Simple Example*

*Backpropagation at time $t$ is hence described by a pair of recursive equations*

- *At time $t$, we compute*

$$\frac{\partial y[t]}{\partial w_{\mathrm{m}}} = w_2 \dot{f}(w_2 h[t]) \frac{\partial h[t]}{\partial w_{\mathrm{m}}}$$

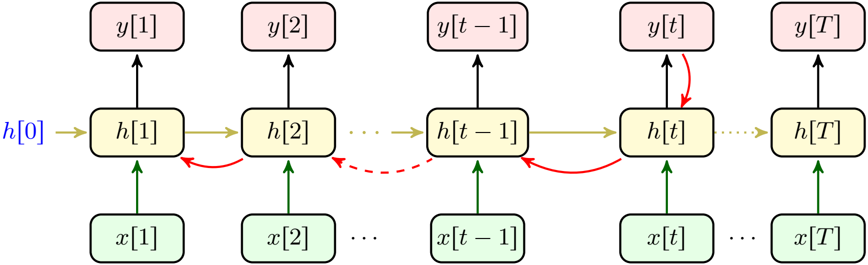
- *For each $i = t, t-1, \ldots, 1$, we use*

$$\frac{\partial h[i]}{\partial w_{\mathrm{m}}} = \dot{f}\left(z[i]\right)\left(h[i-1] + w_{\mathrm{m}} \frac{\partial h[i-1]}{\partial w_{\mathrm{m}}}\right)$$

- *We stop at $i = 1$, where we get*

$$\frac{\partial h[1]}{\partial w_{\mathrm{m}}} = \dot{f}\left(z[1]\right)\left(h[0] + w_{\mathrm{m}} \underbrace{\frac{\partial h[0]}{\partial w_{\mathrm{m}}}}_{0}\right) = \dot{f}\left(z[1]\right) h[0]$$

# Backward Pass Through Time: *Simple Example*




## Key Point

*Propagating back in time is described via a recursive equation*

# Backward Pass Through Time: *Simple Example*

*Recursive equation has an interesting feature*

- *Say we have backpropagated from time $t$*

$$\frac{\partial y[t]}{\partial w_{\rm m}} = w_2 \dot{f}(w_2 h[t]) \frac{\partial h[t]}{\partial w_{\rm m}}$$

  ↳ *We hence have $\partial h[t]/\partial w_{\rm m}, \partial h[t-1]/\partial w_{\rm m}, \ldots, \partial h[1]/\partial w_{\rm m}$*

- *Now, if we want to backpropagate from $t-1$*

  ↳ *We already have $\partial h[t-1]/\partial w_{\rm m}, \partial h[t-2]/\partial w_{\rm m}, \ldots, \partial h[1]/\partial w_{\rm m}$*

  ↳ *We do not need to backpropagate through time anymore*

## Moral of Story

*Pass forward till end of sequence and backpropagate to the beginning just once*

# Backpropagation Through Time (BPTT)

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BPTT():
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1. *Start at* $t$ *with* $\nabla_{\mathbf{W}_\mathrm{m}} \mathbf{y}[t] = \mathbf{W}_2 \circ \dot{f}(\mathbf{W}_2 \mathbf{h}[t]) \circ \nabla_{\mathbf{W}_\mathrm{m}} \mathbf{h}[t]$
2. *Go back in time as* $\nabla_{\mathbf{W}_\mathrm{m}} \mathbf{h}[i] = \dot{f}\left(\mathbf{z}[i]\right) \circ \left(\mathbf{h}[i-1] + \mathbf{W}_\mathrm{m} \circ \nabla_{\mathbf{W}_\mathrm{m}} \mathbf{h}[i-1]\right)$
3. *Stop at* $i = 1$ *with* $\nabla_{\mathbf{W}_\mathrm{m}} \mathbf{h}[1] = \dot{f}\left(\mathbf{z}[1]\right) \circ \mathbf{h}[0]$

\+ *It looks very similar to backpropagation in deep NNs!*

– Exactly! *Even simple RNN is very deep through time*

\+ *Don't we experience vanishing or exploding gradient then!*

– Yes! Let's check it out

# BPTT: *Vanishing Gradient*

*Let's expand the gradient in our example*

$$
\begin{aligned}
\frac{\partial y[t]}{\partial w_{\mathrm{m}}} = \; & w_2 \dot{f}(w_2 h[t]) \dot{f}\left(z[t]\right) h[t-1] \\
& + w_2 \dot{f}(w_2 h[t]) w_{\mathrm{m}} \dot{f}\left(z[t]\right) \dot{f}\left(z[t-1]\right) h[t-2] \\
& + w_2 \dot{f}(w_2 h[t]) w_{\mathrm{m}}^2 \dot{f}\left(z[t]\right) \dot{f}\left(z[t-1]\right) \dot{f}\left(z[t-2]\right) h[t-3] \\
& + \cdots \\
& + w_2 \dot{f}(w_2 h[t]) w_{\mathrm{m}}^{i-1} \left( \prod_{j=0}^{i-1} \dot{f}\left(z[t-j]\right) \right) h[t-i] \\
& + \cdots \\
& + w_2 \dot{f}(w_2 h[t]) w_{\mathrm{m}}^{t-1} \left( \prod_{j=0}^{t-1} \dot{f}\left(z[t-j]\right) \right) h[0]
\end{aligned}
$$

# BPTT: *Vanishing Gradient*

*This says that gradient of hidden state in $i$ steps back in time is multiplied by*

$$w_2 w_{\mathrm{m}}^{i-1} \dot{f}(w_2 h[t]) \left( \prod_{j=0}^{i-1} \dot{f}\left(z[t-j]\right) \right)$$

*Recall deep NNs: we could see two cases*

- *If $\dot{f}(\cdot) > 1$ most of the time*
  - ↳ *very old hidden states can explode the gradient*
  - ↳ *this usually does not happen, because most activations are not like that*
- *If $\dot{f}(\cdot) < 1$ most of the time*
  - ↳ *very old hidden states have pretty much no impact on the gradient*
  - ↳ *this means that the RNN can train only up to a finite memory*
  - ↳ *this typically happens and we call it vanishing gradient through time*

# Handling *Vanishing Gradient Through Time*

+ *Cant we use the same remedy as in deep NNs?*
– Yes and No!

It is important to note that *vanishing gradient through time* is a bit *different*: *we are still updating weights with large enough gradients, but these gradients have no information of long-term memory*

*Solutions to vanishing gradient through time are mainly*

- *Use* $\tanh(\cdot)$ *activation*
  ↳ $\tanh(\cdot)$ *is able to keep memory for a longer period*
- *Use truncated BPTT*
  ↳ *Repeat short-term BPTTs every couple of time intervals*
- *Invoke gating approach*
  ↳ *This is the most sophisticated approach*
  ↳ *It was known for a long time, but received attention much later!*