Applied Deep Learning

Chapter 4: Convolutional Neural Networks

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Components of CNNs

CNNs consist of three major components

1. *Convolutional layers*
   - ↳ *they are the key component*
   - ↳ *they extract features from input by sweeping filters over input*
2. *Pooling layers*
   - ↳ *they smooth the extracted features*
3. *Output FNN*
   - ↳ *they learn label from final extracted features*

We now go through each of these components

Convolution: *2D Arrays*

We already know the definition of convolution for 2D arrays

2D Convolution

Let $\mathbf{X} \in \mathbb{R}^{N \times M}$ *be the input matrix: convolution of* $\mathbf{X}$ *by filter/kernel* $\mathbf{W} \in \mathbb{R}^{F \times F}$ *with stride* S *is denoted by*

$$\mathbf{Z} = \mathrm{Conv}\left(\mathbf{X} | \mathbf{W}, S\right)$$

The matrix $\mathbf{Z}$ *has* $\lfloor (N - F)/S \rfloor + 1$ *rows and* $\lfloor (M - F)/S \rfloor + 1$ *columns and its entry at row* i *and column* j *is computed as*

$$\mathbf{Z}\left[i, j\right] = \mathtt{sum}\left(\mathbf{W} \odot \mathbf{X}_{i,j}\right)$$

with $\mathbf{X}_{i,j}$ *being the corresponding* $F \times F$ *sub-matrix of* $\mathbf{X}$*, i.e.,*

$$\mathbf{X}_{i,j} = \mathbf{X}\left[1 + (i-1)\,S : F + (i-1)\,S, 1 + (j-1)\,S : F + (j-1)\,S\right]$$

Convolution: *2D Arrays*

Let's see an example: *assume* $\mathbf{X} \in \mathbb{R}^{4\times 5}$ *and* $\mathbf{W} \in \mathbb{R}^{2\times 2}$

$$\mathbf{X} = \begin{bmatrix} X_{1,1} & X_{1,2} & X_{1,3} & X_{1,4} & X_{1,5} \\ X_{2,1} & X_{2,2} & X_{2,3} & X_{2,4} & X_{2,5} \\ X_{3,1} & X_{3,2} & X_{3,3} & X_{3,4} & X_{3,5} \\ X_{4,1} & X_{4,2} & X_{4,3} & X_{4,4} & X_{4,5} \end{bmatrix} \circledast \quad \mathbf{W} = \begin{bmatrix} W_{1,1} & W_{1,2} \\ W_{2,1} & W_{2,2} \end{bmatrix}$$

$$\mathbf{X}_{1,1} \longrightarrow \mathtt{sum}(\mathbf{X}_{1,1} \odot \mathbf{W})$$

If we convolve with stride $S = 1$

$$\mathbf{Z} = \begin{bmatrix} Z_{1,1} & Z_{1,2} & Z_{1,3} & Z_{1,4} \\ Z_{2,1} & Z_{2,2} & Z_{2,3} & Z_{2,4} \\ Z_{3,1} & Z_{3,2} & Z_{3,3} & Z_{3,4} \end{bmatrix}$$

Let's verify the dimensions of $\mathbf{Z}$

 # rows = $\lfloor (4-2)/1 \rfloor + 1 = 3$ *# columns* = $\lfloor (5-2)/1 \rfloor + 1 = 4$

Convolution: *Numerical Example*

1	0.9	1.6	0.3	0.4
0.9	0.2	0	0.5	3
0.4	0.2	0.8	0.6	0.3
0.5	0.8	1	0.7	2.1

1	0
1	1

2.1	1.1	2.1	3.8
1.5	1.2	1.4	1.4
1.7	2	2.5	3.4

Convolution: *2D Arrays*

Let's see an example: *assume* $\mathbf{X} \in \mathbb{R}^{4\times 5}$ *and* $\mathbf{W} \in \mathbb{R}^{2\times 2}$

$$\mathbf{X} = \begin{bmatrix} X_{1,1} & X_{1,2} & X_{1,3} & X_{1,4} & X_{1,5} \\ X_{2,1} & X_{2,2} & X_{2,3} & X_{2,4} & X_{2,5} \\ X_{3,1} & X_{3,2} & X_{3,3} & X_{3,4} & X_{3,5} \\ X_{4,1} & X_{4,2} & X_{4,3} & X_{4,4} & X_{4,5} \end{bmatrix} \circledast \quad \mathbf{W} = \begin{bmatrix} W_{1,1} & W_{1,2} \\ W_{2,1} & W_{2,2} \end{bmatrix}$$

$\mathbf{X}_{1,1}$, $\mathbf{X}_{1,2}$, $\mathbf{X}_{2,1}$, $\mathbf{X}_{2,2}$

$$\mathbf{X}_{1,1} \longrightarrow \mathtt{sum}(\mathbf{X}_{1,1} \odot \mathbf{W})$$

Now, we convolve with stride $S = 2$

$$\mathbf{Z} = \begin{bmatrix} Z_{1,1} & Z_{1,2} \\ Z_{2,1} & Z_{2,2} \end{bmatrix}$$

Let's verify the dimensions of $\mathbf{Z}$

 # rows = $\lfloor (4-2)/2 \rfloor + 1 = 2$ *# columns* = $\lfloor (5-2)/2 \rfloor + 1 = 2$

Convolution: *Numerical Example*

1	0.9	1.6	0.3	0.4
0.9	0.2	0	0.5	3
0.4	0.2	0.8	0.6	0.3
0.5	0.8	1	0.7	2.1

2.1	2.1
1.7	2.5

1	0
1	1

Convolution: *Downsampling*

Let's compare the result with strides 1 and 2

2.1		2.1	
1.7		2.5	

2.1	2.1
1.7	2.5

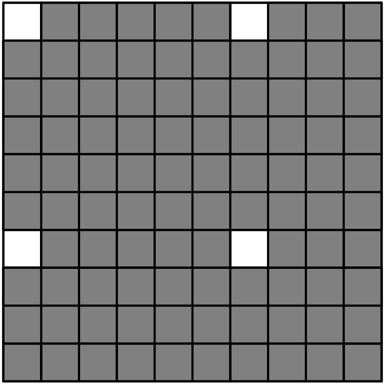
This is downsampling with factor 2!

Downsampling

Downsampling with factor f *drops* the last $f-1$ of *every f columns and rows*

$$\mathrm{dSample}\left(\mathbf{Z}|f\right)$$

Convolution: *Downsampling*



Downsampling with factor 2 $\text{dSample}\left(\mathbf{Z}|2\right)$ *Downsampling with factor 3* $\text{dSample}\left(\mathbf{Z}|3\right)$

Convolution: *Downsampling*

Let's compare the result with strides 1 and 2

2.1		2.1	
1.7		2.5	

2.1	2.1
1.7	2.5

Stride as Downsampling

We can look at stride S as downsampling with factor S, i.e.,

$$\mathrm{Conv}\left(\mathbf{X}|\mathbf{W}, S\right) = \mathrm{dSample}\left(\mathrm{Conv}\left(\mathbf{X}|\mathbf{W}, 1\right)|S\right)$$

Convolution: *Downsampling*

So, we can make an agreement: *by default we consider unit stride, i.e.,* $S = 1$, *i.e., we drop* S *in convolution from now on*

$$\mathrm{Conv}\left(\mathbf{X}|\mathbf{W}\right) = \mathrm{Conv}\left(\mathbf{X}|\mathbf{W}, S = 1\right)$$

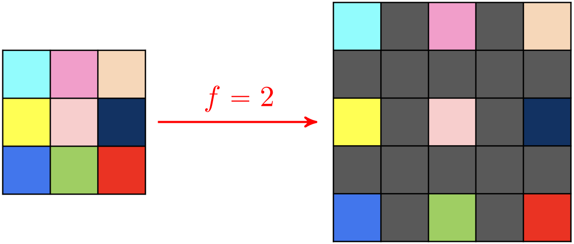
Whenever we need to convolve with stride $S > 1$

We add an downsampling next to the convolution

+ *Why do we do this?*
– We'll see how *easy things get when we want to backpropagate*; however, it also helps to *easily define having a stride smaller than one!*

Convolution: *Upsampling*

We can do the sampling in other way, i.e., add some zeros



Upsampling

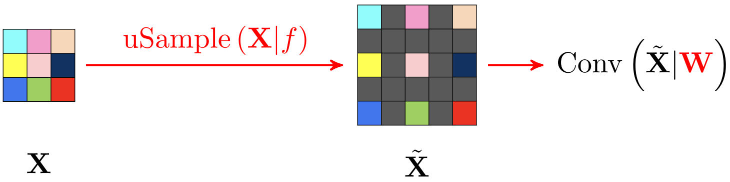
Upsampling with factor f adds $f-1$ rows and columns of zeros after every row and column

$$\mathrm{uSample}\left(\mathbf{Z}|f\right)$$

Convolution *with Fractional Stride*

+ *Where do we use upsampling?*
– If we want to increase the size of the feature map

To increase the size of the feature map, we can do following



This is called fractionally-strided convolution: for $S < 1$ with $1/S$ being integer

$$\mathrm{Conv}\left(\mathbf{X}|\mathbf{W}, S\right) = \mathrm{Conv}\left(\mathrm{uSample}\left(\mathbf{X}|1/S\right)|\mathbf{W}\right)$$

Convolution: *Resampling*

We can extend stride to any fraction $S = S_2/S_1$ with integer S_1 and S_2: *we first do upsampling with factor S_1 and then downsampling with factor S_2*

Moral of Story

Convolution with stride is equivalent with

convolution with resampling

Note that the order of resampling differs

- *Upsampling is done always before convolution*
- *Downsampling is done always after convolution*

The above interpretation of stride helps a lot in backpropagation!

Convolution: *Padding*

In our definition, feature map $\mathbf{Z}$ has *smaller* dimensions than $\mathbf{X}$ *even with stride* $S = 1$. *We can play with dimensions of* $\mathbf{Z}$ *via zero-padding*

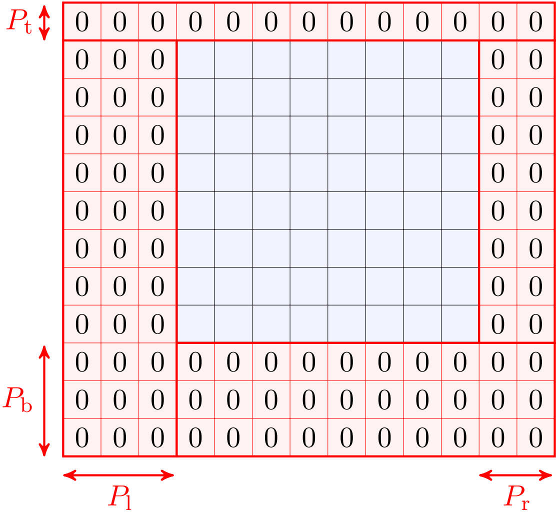
+ *Why should we be interested in changing dimensions of* $\mathbf{Z}$*?*
– We'll see multiple reasons: a simple one is that *we may like to have a same-size feature map to sketch it as an image and compare it to input*

For instance in MNIST, we want to sketch the feature map as a 28 $\times$ *28 image*

$\circledast\ \mathbf{W} \longrightarrow$

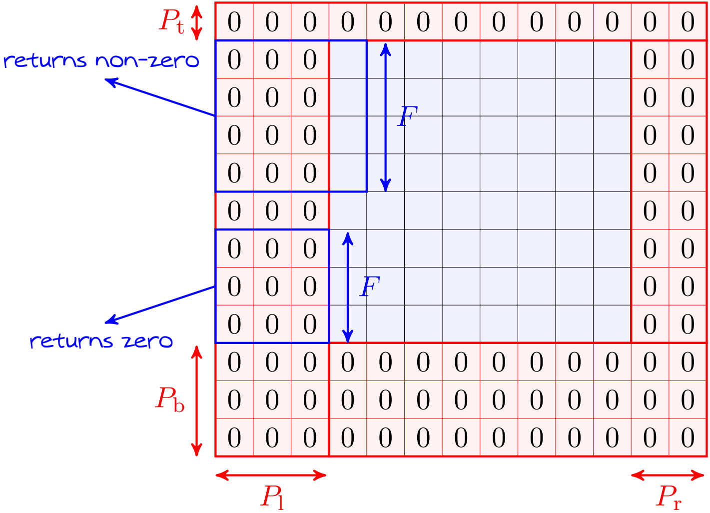
Convolution: *Zero-Padding*

The trick to *resize feature map* is to *pad zeros* at *boundaries*



Convolution: *Zero-Padding*

Typically *we* pad *with widths smaller than filter dimension* F



Convolution: *Zero-Padding*

With zero-padding, the dimensions of feature map can be modified: *say the input has N rows and M columns; then, at the feature map we have*

- $\lfloor (N + P_{\rm t} + P_{\rm b} - F)/S \rfloor + 1$ *rows*
- $\lfloor (M + P_{\rm l} + P_{\rm r} - F)/S \rfloor + 1$ *columns*

In practice, we pad symmetrically, i.e., $P_{\rm b} = P_{\rm t} = P_{\rm r} = P_{\rm l}$

As we see *the dimensions of feature map is a function of*

input dimensions, *stride*, *padding length* and *filter size*

It is thus typical that *we get some of these items not specified*, *e.g.*, *we might get input and output dimensions*, *stride* and *filter size* but not the *padding length*

we can find the padding length from other specifications

Example: *Stride = 2 and Padding*

1	0.9	1.6	0.3	0.4	0
0.9	0.2	0	0.5	3	0
0.4	0.2	0.8	0.6	0.3	0
0.5	0.8	1	0.7	2.1	0

1	0
1	1

2.1	2.1	3.4
1.7	2.5	2.4

Convolution: *Activation*

Convolution is a *spatial linear transform on the input image*

we should activate this linear transform if we go deep

A convolational layer can hence be formally defined as below

Convolutional Layer

Convolutional layer with filter $\mathbf{W} \in \mathbb{R}^{F \times F}$, *bias* b *and activation function* $f(\cdot)$ *transforms the input map* $\mathbf{X}$ *to the activated feature map* $\mathbf{Y}$ *as follows:*

1. *it first applies linear convolution to find* $\mathbf{Z}$

$$\mathbf{Z} = \mathrm{Conv}(\mathbf{X}|\mathbf{W}) + b$$

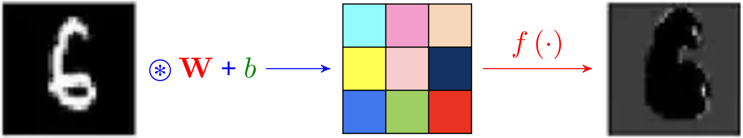
+ applied entry-wise

2. *it then activates* $\mathbf{Z}$

$$\mathbf{Y} = f(\mathbf{Z})$$

$f(\cdot)$ applied entry-wise

Convolution: *Activation*

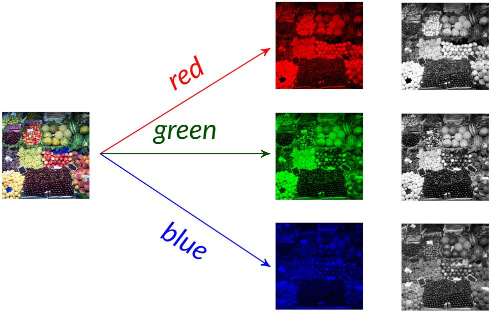


+ *We did not discuss bias before!*
– It's just a scalar *added to all entries*
+ *What kinds of activation are used in CNNs?*
– Similar to FNNs *with ReLU being the popular one*

Convolution: *Multi-Channel Input*

What we discussed up to now *holds for single-channel images: these are gray images that can be represented by 2D pixel arrays*, *e.g., MNIST images*

In practice, we have multi-channel inputs; for instance, *RGB images have three channels: an $N \times M$ color image is stored in the form of three $N \times M$ matrices, one storing red map, another green map, and the other blue map*



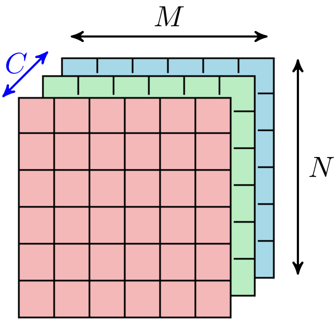
Recap: *3D Tensors*

We can think of multi-channel input as *a tensor of order 3*, *i.e., a 3D array*

Reminder: *Tensor of Order 3*

Tensor $\mathbf{X} \in \mathbb{R}^{C \times N \times M}$ *is a collection of* C *matrices each of size* $N \times M$

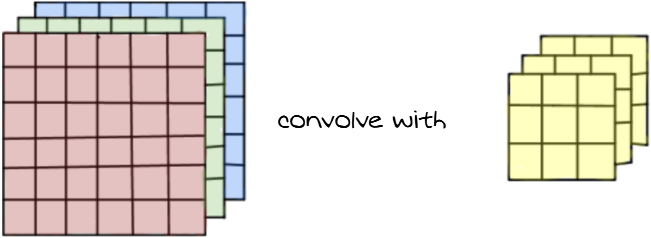
For instance, an $N \times M$ *pixel RGB image is tensor in* $\mathbb{R}^{3 \times N \times M}$



Convolution: *3D Arrays*

+ *How can we extend convolution to these 3D tensors?*
– We can look at 3D tensors as stack of 2D arrays

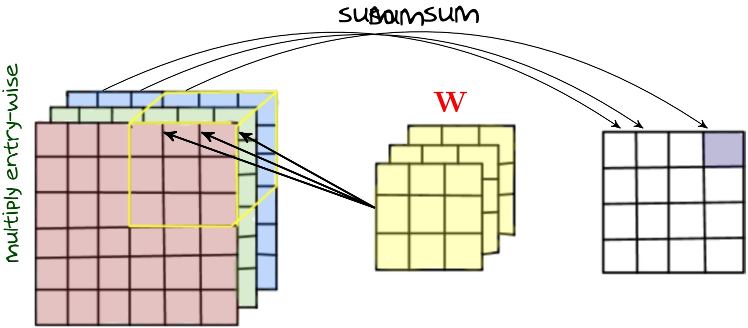
Let's try a visual example: *we want to convolve RGB image with filter* $\mathbf{W}$



Filter should have the same number of channels, i.e., $\mathbf{W} \in \mathbb{R}^{3 \times F \times F}$

Convolution: *General Form*

We convolve every channel of filter with corresponding channel of input, and then sum them up



Convolution: *3D Arrays*

3D Convolution

Convolution of tensor $\mathbf{X} \in \mathbb{R}^{C \times N \times M}$ by filter/kernel $\mathbf{W} \in \mathbb{R}^{C \times F \times F}$ with stride S is denoted by

$$\mathbf{Z} = \mathrm{Conv}\left(\mathbf{X} | \mathbf{W}, S\right)$$

The matrix $\mathbf{Z}$ is a matrix with $\lfloor (N - F)/S \rfloor + 1$ rows and $\lfloor (M - F)/S \rfloor + 1$ columns and its entry at row i and column j is computed as

$$\mathbf{Z}\left[i, j\right] = \mathtt{sum}\left(\mathbf{W} \odot \mathbf{X}_{i,j}\right)$$

with $\mathbf{X}_{i,j}$ being the corresponding $C \times F \times F$ sub-tensor of $\mathbf{X}$, i.e.,

$$\mathbf{X}_{i,j} = \mathbf{X}\left[1 : C, 1 + (i-1)\,S : F + (i-1)\,S, 1 + (j-1)\,S : F + (j-1)\,S\right]$$

3D Convolution: *Summary*

As for 2D arrays, we can *apply stride and by resampling; thus, we write*

$$\mathbf{Z} = \mathrm{Conv}\left(\mathbf{X} | \mathbf{W}\right)$$

from now on and keep in mind that

- $\mathbf{X}$ *is a tensor with C channels*
- $\mathbf{W}$ *is a tensor-like kernel with C channels: some people say with depth C*
- $\mathbf{Z}$ *is a matrix, i.e., it has a single channel*

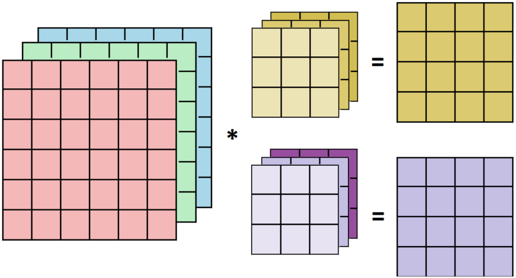
whenever needed we can

- *apply a fractional stride by resampling*
  - *upsampling before* convolution
  - *downsampling after* convolution
- *adjust the size of* $\mathbf{Z}$ *by zero-padding*

Convolution Layer: *General Form*

+ *But, does it make sense to map a large tensor to a small feature map?*
– This is a great point! This is why *we compute multiple feature maps*

A general convolutional layer has *multiple kernels*: *each kernel computes a separate feature map*



Convolution Layer: *General Form*

Multi-Channel Convolutional Layer

Convolutional layer with C-channel input and K-channel output consists of K filters $\mathbf{W}_1, \ldots, \mathbf{W}_K \in \mathbb{R}^{C \times F \times F}$, K biases $b_1, \ldots, b_K$ and an activation function $f(\cdot)$. It transforms the C-channel input $\mathbf{X}$ to the activated K-channel feature tensor $\mathbf{Y}$ as follows:

1. *it finds the feature tensor $\mathbf{Z}$*

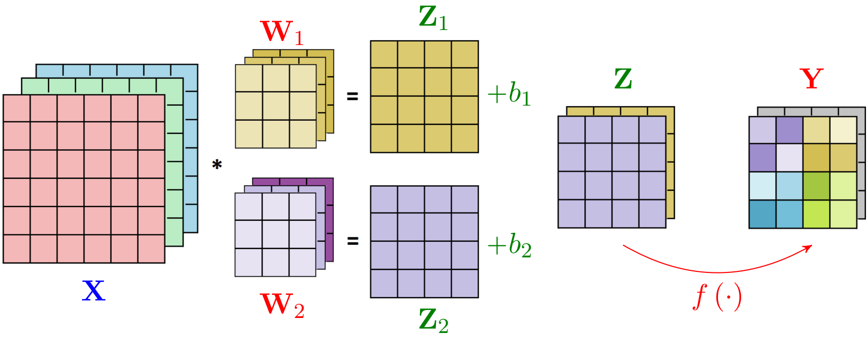
$$\mathbf{Z} = [\mathrm{Conv}(\mathbf{X}|\mathbf{W}_1) + b_1, \ldots, \mathrm{Conv}(\mathbf{X}|\mathbf{W}_K) + b_K]$$

2. *it then activates $\mathbf{Z}$*

$$\mathbf{Y} = f(\mathbf{Z})$$

$f(\cdot)$ applied entry-wise

Multi-Channel Convolution Layer: *Visualization*

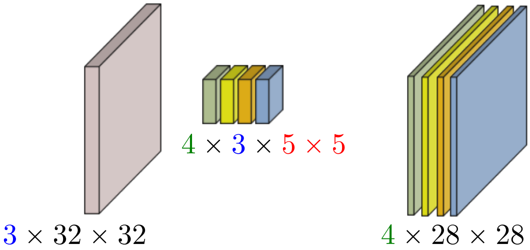


There are few points that we should keep in mind

- *Kernels have the same number of channels (also called depth) as input*
- *Number of kernels equals the number of channels in feature tensor*
- *If* $\mathbf{X}$ *is output of a convolutional layer it may have lots of channels!*
  - ↳ *We should not think that "input has at most 3 channels"!*

How to Read Dimensions of Convolutional Layers

Many details are dropped as they are readily inferred from architecture



In this diagram, we see that $C = 3$ which is the same in kernels and input

- *We have $K = 4$ kernels $\rightsquigarrow$ feature tensor has 4 channels*
- *Kernels have width $F = 5$ $\rightsquigarrow$ we see that $32 - 5 + 1 = 28$*
  - ↳ *stride is one, i.e., $S = 1$*
  - ↳ *no zero-padding is applied $P = 0$*

Idea of Pooling: *Smoothing Filters*

The output of convolution can be jittering which can come from *spatial correlation*: *Pooling acts as a filter by sliding over the extracted features and pooling out a function of each subpart*

- *Pooling can reduce the jittering behavior*
- *It can mix extracted features and potentially improve shift-invariance*

Shift-Invariance

shift-invariance refers to robustness against simple geometric transform of input

+ *How do we do the pooling?*
– There are several pooling techniques but *popular ones are max- and mean-pooling*

Max-Pooling

For max-pooling, we use a filter of size $L \times L$ and slide over the feature map

$$\mathbf{Y} = \begin{bmatrix} Y_{1,1} & Y_{1,2} & \dots & \dots & Y_{1,M} \\ Y_{2,1} & Y_{2,2} & \dots & \dots & Y_{2,M} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ Y_{N,1} & Y_{N,2} & \dots & \dots & Y_{N,M} \end{bmatrix}$$

$\boxed{\mathbf{Y}_{1,1}}$ $\boxed{\mathbf{Y}_{1,2}}$

$$\max \left\{ \mathbf{Y}_{1,1} \mathbf{Y}_{1,2} \mathbf{Y}_{i,j} \right\}$$

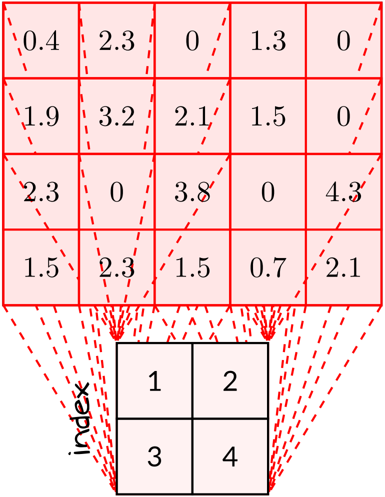
In each window, we pool the maximum

$$\hat{\mathbf{Y}} = \begin{bmatrix} \hat{Y}_{1,1} & \hat{Y}_{1,2} & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \end{bmatrix}$$

Attention

In max-pooling, we also track index of maximizer: we need it in backpropagation

Max Pooling: *Numerical Example*



3.2	3.2	2.1	1.5
3.2	3.8	3.8	4.3
2.3	3.8	3.8	4.3

index

4	3	3	3
2	4	3	4
1	2	1	2

Mean-Pooling

In mean-pooling, we again use an $L \times L$ filter and slide over the feature map

$$\mathbf{Y} = \begin{bmatrix} Y_{1,1} & Y_{1,2} & \dots & \dots & Y_{1,M} \\ Y_{2,1} & Y_{2,2} & \dots & \dots & Y_{2,M} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ Y_{N,1} & Y_{N,2} & \dots & \dots & Y_{N,M} \end{bmatrix}$$

$\mathbf{Y}_{1,1}$ $\mathbf{Y}_{1,2}$

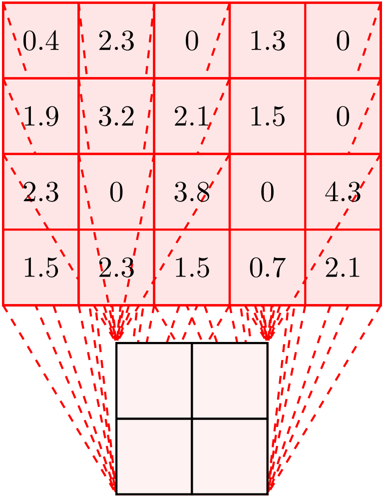
$$\mathtt{mean}\left\{\mathbf{Y}_{1,1}\mathbf{Y}_{1,2}\mathbf{Y}_{i,j}\right\}$$

In each window, we pool the average

$$\hat{\mathbf{Y}} = \begin{bmatrix} \hat{Y}_{1,1} & \hat{Y}_{1,2} & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \end{bmatrix}$$

We can look at mean-pooling as a *convolution* *with* *uniform kernel*

Mean-Pooling: *Numerical Example*



0.4	2.3	0	1.3	0
1.9	3.2	2.1	1.5	0
2.3	0	3.8	0	4.3
1.5	2.3	1.5	0.7	2.1

2.05	1.9	1.225	0.7
1.85	2.275	1.85	1.45
1.525	1.9	1.5	1.775

Mean-Pooling: *Numerical Example*

0.4	2.3	0	1.3	0
1.9	3.2	2.1	1.5	0
2.3	0	3.8	0	4.3
1.5	2.3	1.5	0.7	2.1

$\frac{1}{4}$	$\frac{1}{4}$
$\frac{1}{4}$	$\frac{1}{4}$

2.05	1.9	1.225	0.7
1.85	2.275	1.85	1.45
1.525	1.9	1.5	1.775

mean-pooling $\equiv$ convolution with uniform kernel

Advanced Pooling: *Use a General Function*

We could in general replace max or mean operator with a general function

$$\mathbf{Y} = \begin{bmatrix} Y_{1,1} & Y_{1,2} & \dots & \dots & Y_{1,M} \\ Y_{2,1} & Y_{2,2} & \dots & \dots & Y_{2,M} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ Y_{N,1} & Y_{N,2} & \dots & \dots & Y_{N,M} \end{bmatrix}$$

$\mathbf{Y}_{1,1}$ $\mathbf{Y}_{1,2}$

$$\Pi\left(\cdot\right) : \mathbb{R}^{L \times L} \mapsto \mathbb{R}$$

Typical functions are norms, e.g., ℓ_2-norm

$$\hat{\mathbf{Y}} = \begin{bmatrix} \hat{Y}_{1,1} & \hat{Y}_{1,2} & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \end{bmatrix}$$

We can even replace it with a small NN!

Pooling with Stride

Similar to convolution, *we can apply pooling with stride*

- *We usually do not use fractional stride with pooling*
  - ↳ *Upsampling the input, only adds zeros*
  - ↳ *Extra zeros usually do not make any gain: think of max-pooling for instance*
- *Integer stride can be seen as downsampling*
  - ↳ *We apply pooling with unit stride*
  - ↳ *We down-sample output with sampling factor = stride*
- *In practice, we often leave integer strides for pooling layer*
  - ↳ *We apply convolution with unit stride*
  - ↳ *If we need to down-sample, we do it later in the pooling layer*

Pooling: *Few Remarks*

+ *Do we always pool after convolution?*
– Not really! In many architectures pooling is applied every couple of convolutional layers

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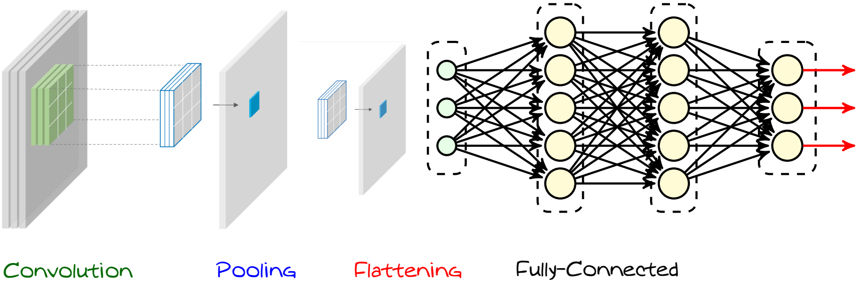
It is worth noting that

- *In many architectures pooling is applied every couple of convolutional layers*
  ↳ *We apply multiple convolutional layers*
  ↳ *We then apply a pooling layer with stride*
- *Most poolings have no learnable parameters; thus, they are cheap*
  ↳ *Max and mean-pooling have no weights*
- *Filter size for pooling can be different from the convolutional layers*
  ↳ *They are typically in the same range*

Output FNN: *Flattening*

Let's recall our simple CNN

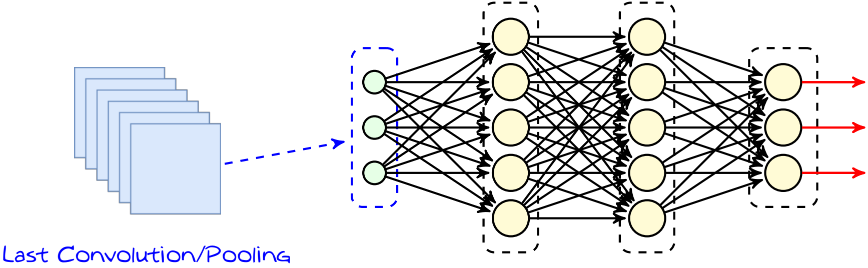


Once we are over with convolution and pooling we flatten final feature tensor

Flattening

In flattening, we sort all entries of the feature tensor into a vector

Flattening



Say we have a *feature tensor* with K channels

↳ *each channel is an $N \times M$ map after last convolution or pooling*

We then have an input to the FNN with NMK entries

After flattening everything goes as before through the fully-connected FNN